MATH 579 S26, Exam 1 Solutions

1. A hand in the card game bridge consists of a set of 13 cards, from the standard 52-card deck. What is the probability that a hand contains none of {10,J,Q,K,A}?

The number of such cards is the number of functions from $\{\bullet^{13}\}$ to $\{2,3,\ldots,9\}\times\{\clubsuit,\diamondsuit,\heartsuit,\spadesuit\}$, which by the twelvefold way is $\binom{8\times 4}{13}=\frac{32^{\underline{13}}}{13!}$. The number of bridge hands in total is the number of functions $\{\bullet^{13}\}$ to $\{2,3,\ldots,9,10,J,Q,K,A\}\times\{\clubsuit,\diamondsuit,\heartsuit,\spadesuit\}$, which by the twelvefold way is $\binom{13\times 4}{13}=\frac{52^{\underline{13}}}{13!}$. Hence the probability is $\frac{32^{\underline{13}}}{52^{\underline{13}}}$.

Story time: This probability turns out to be approximately $\frac{1}{1828}$. The Earl of Yarborough, in the 19th century, regularly bet people at odds of 1000-to-1 against getting such a terrible hand. As you can see, the odds were very much in his favor. For this reason, these hands are called Yarboroughs.

2. We consider words from the five-letter alphabet {A,B,C,D,E}. How many seven-letter words are there that contain exactly one double letter?

We build such words in two steps.

First, we build six-letter words (from left to right) that contain no double letters. For the first letter, there are five choices. For each remaining letter, there are always four choices (any letter except the one we just chose). Hence there are $5\cdot 4^5$ such six-letter words.

Next, we choose one of the six letters in the word we built, to duplicate into a double letter. For example, ABCABC could become AABCABC or ABBCABC or ... or ABCABCC. Hence each of the $5\cdot 4^5$ words from the previous step leads to six words of the type we want. The total number is therefore $6\cdot 5\cdot 4^5$.

[This happens to equal 30720, but you need not calculate such a big number by hand.]

3. We consider words from the five-letter alphabet {A,B,C,D,E}. How many seven-letter words are there that do NOT use all five letters from the alphabet?

We will count seven-letter words that DO use all five letters from the alphabet, and use the sum principle to subtract this from the total number of seven-letter words.

To count the first type of words, we consider a word as a function from $\{1,2,3,4,5,6,7\}$ to $\{A,B,C,D,E\}$ (i.e., if $f(3)=A$, then the third letter is A), and we want this to be surjective. By the twelvefold way there are $5!\left\{{7 \atop 5}\right\}=120\cdot 140$ such words.

To count the second type of words, we again count these same functions, no longer requiring them to be surjective. By the twelvefold way there are 5^7 such words.

Hence, the desired answer is $5^7-120\cdot 140$.

[This happens to equal $78125-16800=61325$, but you need not calculate this.]

4. Consider the Mersenne numbers $M_i=2^i-1$, for $i\geq 1$. Prove that some Mersenne number is divisible by 75.

Consider the first 76 Mersenne numbers (we could do it with 75, with some extra care, but why bother?) and their remainders upon division by 75. By the PHP, at least two of them must have the same remainder, i.e. there are $i,j\in[76]$, with $i>j$, such that $M_i\equiv M_j \pmod{75}$. Hence $M_i-M_j\equiv 0 \pmod{75}$, so $75|(M_i-M_j)$. We observe that $M_i-M_j=(2^i-1)-(2^j-1)=2^i-2^j=2^j(2^{i-j}-1)$. Hence $75|2^j(2^{i-j}-1)$. Since $\gcd(75,2^j)=1$, we must have $75|2^{i-j}-1$, so $75|M_{i-j}$. Since $i,j\in[76]$ and $i>j$, in fact $i-j\in[76]$, so the proof is complete.

In case you're curious, M_{20} is the first Mersenne number divisible by 75.

5. Seven points are chosen from a regular hexagon with side length 1cm. Prove that some pair of these points must be within distance 1cm of each other.

Divide the hexagon into six equilateral triangles, like slicing a pizza. By the PHP, some triangle must have at least two of the seven points (points on a boundary may be considered in either triangle). Those two points in the same equilateral triangle must be within 1cm of each other, since that is the greatest possible distance in the triangle.

6. How many solutions are there to the equation $x_1 + x_2 + x_3 = 31$, where x_1 is a nonnegative even integer, x_2 is a positive even integer, and x_3 is a positive odd integer?

We write $x_1 = 2a, x_2 = 2b + 2, x_3 = 2c + 1$, where $a, b, c \in \mathbb{N}_0$ (nonnegative integers). Substituting, we get $2a + (2b + 2) + (2c + 1) = 31$, so $2a + 2b + 2c = 28$, so $a + b + c = 14$. Hence each solution that we seek corresponds to a weak composition of 14 into three parts. By exercise 1.15, this is counted by $\left(\!\binom{3}{14}\!\right) = \binom{16}{2} = \frac{16^{\underline{2}}}{2!} = \frac{16\cdot 15}{2} = 8 \cdot 15 = 120$.

7. Find, with proof, a simple formula for $\left\{ {a \atop 2} \right\}$ $(a \geq 2)$.

We need to partition $[a] = \{1, 2, \ldots, a\}$ into exactly two parts. Let's focus on the part that contains 1. In addition to 1, it contains some subset of $\{2, \ldots, a\}$, but not everything (since that would leave no elements for the other part).

The number of subsets of $\{2, \ldots, a\}$ can be counted by functions from $\{2, \ldots, a\}$ to {in,out}. The twelve-fold way tells us the number of such functions is 2^{a-1}. However, one of these functions sends everything to "in". We exclude it, using the sum principle. Therefore, the answer to the problem is $\left\{ {a \atop 2} \right\} = 2^{a-1} - 1$.

8. Let $b \in \mathbb{Z}$. Prove that $\binom{a}{b} = (-1)^b\binom{b-a-1}{b}$.

Fans of algebraic calculations will enjoy this one. Note that this can't be done with the $\frac{a!}{b!(a-b)!}$ formula, in fact the question doesn't even make sense if this is the only tool you have. Suppose that $b \geq 0$.

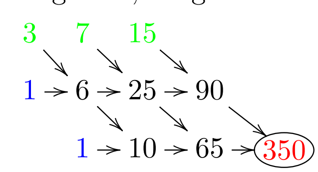
$(-1)^b\binom{b-a-1}{b} = (-1)^b\frac{(b-a-1)^{\underline{b}}}{b!} = (-1)^b\frac{(b-a-1)(b-a-2)\cdots(b-a-b)}{b!} = \frac{(a+1-b)(a+2-b)\cdots(a+b-b)}{b!} =$ (reverse order) $\frac{(a+b-b)\cdots(a+2-b)(a+1-b)}{b!} = \frac{a\cdots(a-(b-2))(a-(b-1))}{b!} = \frac{a^{\underline{b}}}{b!} = \binom{a}{b}$.

If instead $b < 0$, then both sides are 0, by definition of binomial coefficients.

9. Calculate, with justification, $\left\{ {7 \atop 4} \right\}$. You may not use the table in the back of your coursepack, but you may use any formulas proved in the exercises.

METHOD 1: By exercise 1.19, $\left\{ {a \atop a} \right\} = 1$ and $\left\{ {a \atop 2} \right\} = 2^{a-1} - 1$. Now we apply exercise 1.26 repeatedly until we get to one of these boundary values: $\left\{ {4 \atop 3} \right\} = 3\left\{ {3 \atop 3} \right\} + \left\{ {3 \atop 2} \right\}$, $\left\{ {5 \atop 3} \right\} = 3\left\{ {4 \atop 3} \right\} + \left\{ {4 \atop 2} \right\}$, $\left\{ {6 \atop 3} \right\} = 3\left\{ {5 \atop 3} \right\} + \left\{ {5 \atop 2} \right\}$, $\left\{ {5 \atop 4} \right\} = 4\left\{ {4 \atop 4} \right\} + \left\{ {4 \atop 3} \right\}$, $\left\{ {6 \atop 4} \right\} = 4\left\{ {5 \atop 4} \right\} + \left\{ {5 \atop 3} \right\}$, $\left\{ {7 \atop 4} \right\} = 4\left\{ {6 \atop 4} \right\} + \left\{ {6 \atop 3} \right\}$.

Putting it all together, we get 350.



METHOD 2: By exercise 1.26, $\left\{ {7 \atop 4} \right\} = 4\left\{ {6 \atop 4} \right\} + \left\{ {6 \atop 3} \right\}$. By exercise 1.o, $\left\{ {6 \atop 4} \right\} = \left\{ {6 \atop 6-2} \right\} = \frac{6(6-1)(6-2)(3\cdot 6-5)}{24} = \frac{6\cdot 5\cdot 4\cdot 13}{24} = 5 \cdot 13 = 65$. By exercise 1.n, $\left\{ {6 \atop 3} \right\} = \frac{3^{6-1}+1}{2} - 2^{6-1} = \frac{3^5+1}{2} - 2^5 = 90$.

Putting it all together, $\left\{ {7 \atop 4} \right\} = 4 \cdot 65 + 90 = 350$.

10. Find each partition of 10 into exactly four parts, identify any self-conjugate ones, and draw the Ferrers diagrams for those self-conjugate ones.

There are nine such partitions, presented here in decreasing lex order.

7+1+1+1   6+2+1+1   5+3+1+1   5+2+2+1   (4+4+1+1)
(4+3+2+1)   (4+2+2+2)   3+3+3+1   3+3+2+2

Note that any self-conjugate partition into 4 parts must have its biggest part exactly 4 (four rows, hence four columns). Testing the three candidates (circled) gives only 4+3+2+1 as self-conjugate. Here is its Ferrers diagram.